On Yamamuro's inverse and implicit function theorems in terms of calibrations

by

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Abstract. For the Fréchet space $E = C^{\infty}(\mathrm{S}^{1\cdot})$ and for a smooth $\varphi : I\!R \to I\!R$, we prove that the associated map $E \to E$ given by $x \mapsto \varphi \circ x$ satisfies the continuous $\mathrm{B}\Gamma$–differentiability condition in Yamamuro's inverse function theorem only if φ is affine. Via more complicated examples, we also generally discuss the importance of testing the applicability of proposed inverse and implicit function theorems by this kind of simple maps.

In [7; p. 3], we mentioned that in [10] quite special differentiabilities are designed hoping to get inverse and implicit function theorems (see [10; 5.2, 5.3, p. 45]) applicable to maps of Fréchet function spaces. Our Theorem 11 together with the examples and remarks below indicates this hope to be overoptimistic. Before getting into the proof of Theorem 11 in B below, we discuss the general relevance of this kind of results to refuting applicability of proposed inverse or implicit function theorems, shortly IFTs. For the notations neither immediately guessed by the reader nor explained below, we refer to [7; pp. 4 − 6] and [4; pp. 4 − 9].

A. Introductory and motivating considerations

We first note that inverse (inFT) and implicit (imFT) function theorems are to some extent complementary parts of more general IFT type results. Assume that we are given a class $\mathcal{C}_1$ of differentiable maps of a certain order, and loosely say that a function f is *regular* iff we have $(E, F, f) \in \mathcal{C}_1$ for some implicitly understood spaces E, F. Note that $u`z$ is the function value of u at z, which conventionally is denoted by "$u(z)$". We also have $f^{-\iota}``B = \{ x : \exists\, y \in B \,;\, (x, y) \in f \}$.

Now, first suppose that we have an imFT for functions $f \subseteq A \times B \times B$ where A and B are subsets of structured (e.g. topological/locally convex/normed) vector spaces E and F, respectively. The imFT says that under suitable conditions for given $(x_0, y_0, b) \in f$, there is a regular g with $(x_0, y_0) \in g \subseteq f^{-\iota}``\{b\}$, and hence we have $f`(x, g`x) = b$ for all $x \in \mathrm{dom}\, g$. Suppose further that we have a function h with $(y_0, x_0) \in h \subseteq B^{\times 2\cdot} = B \times B$ and that we would be pleased with getting a regular g with $(x_0, y_0) \in g \subseteq h^{-\iota}$, a local right inverse to h. If suitable conditions are satisfied, in the imFT we may take $E = F$ and $b = \mathbf{0}_E$ and f given by the prescription $(x, y) \mapsto x - h`y$, and get g as required.

Conversely, suppose we have an inFT and that for a given function f we want to establish an implicit function g as above. Then (under suitable conditions) with

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$B_1 = A \times B$ we may take $h = [\,\mathrm{pr}_1\,, f\,]_{\mathrm{f}} \subseteq B_1{}^{\times 2\cdot}$ given by $(x\,,y) \mapsto (x\,, f\,\grave{}\,(x\,,y))$ in the inFT obtaining $g_1 \subseteq h^{-\iota}$, and finally get $g = \mathrm{pr}_2 \circ g_1 \circ [\,\mathrm{id}\,, \mathbf{U} \times \{b\}\,]_{\mathrm{f}}$ given by the prescription $x \mapsto \mathrm{pr}_2 \circ g_1\grave{}\,(x\,,b)$.

1 *Remarks***.** As formulated above, from a local imFT one can only get a local right inFT. However, usually one has such a topological situation that existence of some W is guaranteed so that in the imFT we may take $g = f^{-\iota}$``$\{b\} \cap W$ with W a neighbourhood of $(x_0\,,y_0)$ in the product topology, cf. [4; Sec. 4, Theorems 1, 5, pp. 19, 20]. From this stronger formulation of an imFT we get a "two-sided" local inFT where existence of V is guaranteed such that $y_0 \in V$ and $h\,|\,V$ injective and regular with also $(h\,|\,V\,)^{-\iota}$ regular. See, e.g. [4; Corollary 4.6, p. 22].

Further, it should be noted that proving an imFT directly may permit one to get a result more general than one would get from a previous inFT via our observations above, and similarly with the roles reversed. For example, considering the classical Banach or normed space calculus, if one first gets the inFT, one must consider maps $(E\,,E\,,h)$ where E is a *complete* normed space. From this inFT one can only get an imFT for functions $f \subseteq A \times B \times B$ where also the "parameter" set A lies in a Banach space. As for the converse situation, from our imFT in [3; p. 235] we can directly get only the classical inFT.

To make the preceding more concrete, we next consider some examples. For the definition of the Gateaux derivative function ${}^{\mathrm{rGat}}D_{EF}\,f$ of a map $(E\,,F\,,f\,)$ see Definitions 7 below. Also note that $1\boldsymbol{.} = \{\emptyset\}$ and $2\boldsymbol{.} = \{\emptyset\,, 1\boldsymbol{.}\}$ and $\{\,i^{\bullet} : i \in I\!N_{\mathrm{o}}\,\} = Z\!\!\!Z_+ \subseteq I\!R$ and $\{\,n\boldsymbol{.} : n \in Z\!\!\!Z_+\,\} = I\!N_{\mathrm{o}}$ with $(n\boldsymbol{.})^{\bullet} = n$ and $(i^{\bullet})\boldsymbol{.} = i$ and $i^+ = i + 1\boldsymbol{.} = i \cup \{\,i\,\}$ and $i^{++} = (i^+)^+$ for $i \in I\!N_{\mathrm{o}}$ and $n \in Z\!\!\!Z_+$.

2 *Example***.** Existence and "regular" dependence on parameters (including initial/boundary values and the "equations" themselves) of solutions to partial differential equations can be obtained by using IFTs. To get a simple particular case of this general and vague scheme, we consider a partial problem of the more general one already solved in [4; Section 5, Example 7, Theorem 8, pp. 30–31].

Namely, let $I = [\,0\,,1\,]$ and $Q = I \times I\!R$, and let the fixed smooth $\varphi : Q \times I\!R \to I\!R$ be such that $\varphi\,\grave{}\,(t\,,\eta + 1\,,\xi) = \varphi\,\grave{}\,(t\,,\eta\,,\xi)$ holds for all $(t\,,\eta\,,\xi) \in Q \times I\!R$. Letting $E = C^{\infty}_{\mathrm{per}}(I\!R)$, see the few lines just before Lemma 8 below, and with

$$S = \{\,x : \forall\, t \in I\,, \eta \in I\!R\,;\; x\,\grave{}\,(t\,,\eta + 1) = x\,\grave{}\,(t\,,\eta)\,\} \qquad \text{also letting}$$

$F_0 = C^{\,1\cdot}(Q)_{/\,S}$ and $F = C^{\,\infty}(Q)_{/\,S}$, assume that $x_0 \in \upsilon_s E$ and $y_0 \in \upsilon_s F$ with

$$\partial_1\, y_0 + \partial_2\, y_0 = \varphi \circ [\,\mathrm{id}\,, y_0\,]_{\mathrm{f}} \text{ and } y_0(0\,,\cdot) = x_0\,.$$

In other words, we have a simple nonlinear partial differential equation on a compact cylinder with boundary values x_0 specified on one of the two components. We are interested to know (?) whether there is an open neighbourhood U of x_0 in the space E such that for every $x \in U$ there is a unique solution $y \in \upsilon_s F_0$, and that in fact $y \in \upsilon_s F$ and also this correspondence $x \mapsto y$ defines a smooth map $g : E \to F$.

From [4; Theorem 5.8] it follows that the answer to (?) is affirmative. However, for the purposes of the note at hand, we sketch another approach via an inFT as follows. As we saw in [4; Example 5.5, pp. 26–27], we can write the equation in question for the unknown y with "initial" values x as $y = \mathrm{S}\,x + \mathrm{I}\,(\varphi \circ [\,\mathrm{id}\,, y\,]_{\mathrm{f}})$, where $\mathrm{S}\,x = \langle\, x\,\grave{}\,(\eta - t) : \zeta = (t, \eta) \in Q\,\rangle$ and $\mathrm{I}\,v = \big\langle\, \int_0^{\sigma_{\mathrm{rd}}\zeta} v \circ ?_\tau\grave{}\,\zeta \;\mathrm{d}\,\tau : \zeta \in Q\,\big\rangle$ with $?_\tau = \langle\,(\tau, \eta - t + \tau) : \zeta = (t, \eta) \in Q\,\rangle$.

Now, if with $G = E \sqcap F$ and $f = \langle\, y - \mathrm{S}\,x - \mathrm{I}\,(\varphi \circ [\,\mathrm{id}\,, y\,]_{\mathrm{f}}) : z = (x, y) \in \upsilon_s G\,\rangle$ and $h = [\,\mathrm{pr}_1\,, f\,]_{\mathrm{f}}$ and $\tilde{h} = (G, G, h)$, we have an inFT applicable to the map $\tilde{h}$,

we can get a local *solution map* (E, F, g) as explained above. However, this does not give uniqueness of the solution. Moreover, we shall see in Remarks 5(b) below that at least the inFT [10; 5.2, p. 45] *is not* applicable in the generic case where the equation is not linear near y_0, i.e. when no $\varepsilon \in I\!R^+$ exists with $\partial_3^{\,2\cdot}\varphi``N \subseteq \{0\}$ for $N = Q \times I\!R \cap \{(\zeta, \xi) : \exists \xi_1 ; (\zeta, \xi_1) \in y_0 \text{ and } |\xi - \xi_1| < \varepsilon\}$.

3 *Example.* Let $\varepsilon_0 \in I\!R^+$ and $N \in I\!N$, and let $Q_0 =]-\varepsilon_0, \varepsilon_0[\times \bar{\Omega}$ where $\bar{\Omega}$ is the closure of a bounded open Ω contained in $I\!R^N$. Let $A \subseteq 1. \times I\!N_o^N$ be finite, and let $\varphi_{\text{old}} : Q_0 \times I\!R^A \to I\!R$ be smooth. For $\bar{\alpha} = (i, \alpha) \in I\!N_o \times I\!N_o^N$ and for a smooth function $y : \bar{\Omega}(T) = [0, T] \times \bar{\Omega} \to I\!R$ with $T \in I\!R^+$, now the iterated partial derivative $\partial^{\bar{\alpha}} y = \partial_T^i \partial_S^\alpha y$ is defined in an "obvious" manner. Letting the jet $\mathrm{J}\, y : \bar{\Omega}(T) \to I\!R^A$ be defined by $\zeta = (t, \eta) \mapsto \langle \partial^{\bar{\alpha}} y`\zeta : \bar{\alpha} \in A \rangle$, we may consider the partial differential equation $\mathrm{oldE}(y_{\text{ini}}, y_{\text{old}}) : \partial_T^{1\cdot} y_{\text{old}} = \varphi_{\text{old}} \circ [\,\mathrm{id}, \mathrm{J}\, y_{\text{old}}\,]_{\text{f}}$ with initial condition $y_{\text{old}}(0, \cdot) = y_{\text{ini}}$.

Assume further that we are given suitable boundary conditions in the form of a linear subspace S_0 in the Fréchet space $C^\infty(\bar{\Omega})$. Putting $E_{\text{ini}} = C^\infty(\bar{\Omega})_{/S_0}$ and $\mathrm{F}(T) = C^\infty(\bar{\Omega}(T))_{/S}$, where $S = I\!R^{\bar{\Omega}(T)} \cap \{y : \forall t \in [0, T] ; y(t, \cdot) \in S_0\}$, for a given $y_{\text{ini}0} \in \upsilon_s E_{\text{ini}}$ suppose that we are interested to know (?) whether some ε with $0 < \varepsilon < \varepsilon_0$ and an open neighbourhood U of $y_{\text{ini}0}$ in E_{ini} and a smooth function $g : E_{\text{ini}} \supseteq U \to \mathrm{F}(\varepsilon)$ exist with $g \subseteq \{(y_{\text{ini}}, y_{\text{old}}) : \mathrm{oldE}(y_{\text{ini}}, y_{\text{old}})\}$.

In other words, we want to know whether our initial-boundary value problem is in a certain sense "locally well-posed". We approach the problem via an inFT as follows. Let $E = \boldsymbol{R} \sqcap E_{\text{ini}}$ and $F = \mathrm{F}(1)$ and $G = E \sqcap F$. Also let $\mathrm{S}\, y_{\text{ini}} = y_{\text{ini}} \circ \mathrm{pr}_2 | \bar{\Omega}(1)$ and $\mathrm{J}_0\, y = \langle \int_0^t y`(\tau, \eta)\, \mathrm{d}\tau : \zeta = (t, \eta) \in \bar{\Omega}(1) \rangle$. Considering the family $\varphi = \langle \langle T \varphi_{\text{old}}`(T t, \eta, \xi) : P = (t, \eta, \xi) \in \bar{\Omega}(1) \times I\!R^A \rangle : T \in]-\varepsilon_0, \varepsilon_0[\rangle$, if with $0 < \varepsilon < \varepsilon_0$ we let $\iota = \langle \langle y`(\varepsilon^{-1} t, \eta) : \varepsilon : \zeta = (t, \eta) \in \bar{\Omega}(\varepsilon) \rangle : y \in \upsilon_s F \rangle$, then ι is a linear homeomorphism $F \to \mathrm{F}(\varepsilon)$ such that for $y \in \upsilon_s F$ satisfying the equation $y = \mathrm{S}\, y_{\text{ini}} + \mathrm{J}_0(\varphi`\varepsilon \circ [\,\mathrm{id}, \mathrm{J}\, y]_{\text{f}})$ also $\mathrm{oldE}(y_{\text{ini}}, \iota`y)$ holds.

Consequently, we get (?) affirmatively answered if with

$$W_0 = (0, y_{\text{ini}0}, \mathbf{0}_F, z_0), \quad \text{where } z_0 = (x_0, y_0) = (0, y_{\text{ini}0}, \mathrm{S}\, y_{\text{ini}0}), \text{ and}$$
$$h = \langle (x, y - \mathrm{S}\, y_{\text{ini}} - \mathrm{J}_0(\varphi`T \circ [\,\mathrm{id}, \mathrm{J}\, y]_{\text{f}})) : z = (x, y) = (T, y_{\text{ini}}, y) \in \upsilon_s G \text{ and } |T| < \varepsilon_0 \rangle$$

we show existence of h_1 with $W_0 \in h_1 \subseteq h^{-\iota}$ and (G, G, h_1) smooth. Namely, then there are $\varepsilon \in I\!R^+$ and an open neighbourhood U of $y_{\text{ini}0}$ in E_{ini} such that we have $[-\varepsilon, \varepsilon] \times U \times \{\mathbf{0}_F\} \subseteq \mathrm{dom}\, h_1$, and we may take $g = \iota \circ g_1$ where g_1 is defined by the prescription $U \ni y_{\text{ini}} \mapsto \mathrm{pr}_2 \circ h_1`(\varepsilon, y_{\text{ini}}, \mathbf{0}_F)$.

We shall see in Remarks 5(c) below that [10; 5.2, p. 45] is of no use here provided the boundary conditions satisfy $\upsilon_s C(\bar{\Omega}) \cap \{v : v | \Omega \in \upsilon_s \mathcal{D}(\Omega)\} \subseteq S_0$, and the "equation" or the pair $\boldsymbol{\varepsilon} = (\varphi_{\text{old}}, y_{\text{ini}0})$ is *initially strictly nonlinear*, meaning that some $\eta_0 \in \Omega$ exists such that for every ε with $0 < \varepsilon \leq \varepsilon_0$ there are $\bar{\alpha}, \bar{\alpha}_1 \in A$ and $t > 0$ and $\eta \in \Omega$ and $\xi \in I\!R^A$ with $t + |\eta - \eta_0|_\Sigma + |\xi - \mathrm{J}\,\mathrm{S}\, y_{\text{ini}0}`(t, \eta)|_\Sigma < \varepsilon$ and $\partial_{\bar{\alpha}} \partial_{\bar{\alpha}_1} \varphi_{\text{old}}`(t, \eta, \xi) \neq 0$, generally letting $|\zeta|_\Sigma = \sum \langle |\zeta`i| : i \in \mathrm{dom}\, \zeta \rangle$.

Note that according to the preceding definition from $\boldsymbol{\varepsilon}$ not being initially strictly nonlinear it follows existence of N_2 with the property $\mathrm{J}\,\mathrm{S}\, y_{\text{ini}0} | (\{0\} \times \Omega) \subseteq N_2 \in \tau_{\text{rd}}(\boldsymbol{R} \sqcap \boldsymbol{R}^{N]_{\text{tvs}}} \sqcap \boldsymbol{R}^{A]_{\text{tvs}}}) \cap (I\!R_+ \times \Omega \times I\!R^A)$, and also such that for every fixed $\zeta \in \mathrm{dom}\, N_2$ there is an affine $\alpha : \mathbf{R}^{A]_{\text{vs}}} \to \mathbf{R}$ with $\varphi_{\text{old}} | N_2(\zeta, \cdot) \subseteq \alpha$. That is, then the equation is in a sense locally linear near the initial values.

4 *Example.* Letting

$[x]_{\mathrm{S1}} = \{\, y : x, y \in D_0 \text{ and } \exists\, n \in \mathbb{Z}\,;\, \forall\, s \in \mathbb{R}\,;\, x`s = y`s + n \,\}$, where

$D_0 = \upsilon_s C^{\infty}(\mathbb{R}) \cap \{\, x : 0 \notin \operatorname{rng} \mathrm{D}\, x \text{ and } \forall\, s \in \mathbb{R}\,;\, x`(s+1) = x`s + 1 \,\}$,

with $D = \{\, [x]_{\mathrm{S1}} : x \in D_0 \,\}$ also putting

$$\gamma = D^{\times 3\cdot} \cap \{\, (\hat{x}, \hat{y}, \hat{z}) : \exists\, x, y, z\,;\, (x, y, z) \in \hat{x} \times \hat{y} \times \hat{z} \text{ and } z = y \circ x \,\},$$

then γ is a group(operation) on D. Also let $\iota = \mathrm{id}\,\mathbb{R}$ and $E = C^{\infty}_{\mathrm{per}}(\mathbb{R})$ and $U_1 = \upsilon_s C^{1\cdot}_{\mathrm{per}}(\mathbb{R}) \cap \{\, x : |\, x`0 \,| < \frac{1}{2} \text{ and } -1 \notin \operatorname{rng} \mathrm{D}\, x \,\}$ and $U = \upsilon_s E \cap U_1$ and

$$\phi = \langle\, [\, u + \iota \,]_{\mathrm{S1}} : u \in U \,\rangle^{-\iota}.$$

Considering the smoothness $\mathcal{S} = C^{i\cdot\infty}_{\Pi}(\boldsymbol{R})$ corresponding to the differentiability class $C^{\infty}_{\Pi}(\boldsymbol{R}) = C^{\infty}_{\mathrm{c}}(\boldsymbol{R})$, by a result similar to [1; Ch. 3, Sec. 1.9, Proposition 18, pp. 226−227] and [2; Proposition 1.13, pp. 354−355], if certain conditions $(*)$ hold, there is a unique M with (M, γ) a Lie $\mathcal{S}$−group such that for some V with $[\,\iota\,]_{\mathrm{S1}} \in V$ also $M \cup \{\, (\phi \,|\, V, E) \,\}$ is an $\mathcal{S}$−atlas.

One of the conditions $(*)$ holds if for $g = U^{\times 3\cdot} \cap \{\, (x, y, z) : y = z + x \circ (\iota + z) \,\}$ and $\tilde{g} = (E \sqcap E, E, g)$ we have $\tilde{g} \in C^{\infty}_{\Pi}(\boldsymbol{R})$. Letting

$$f = \{\, (x, y, z, y - z - x \circ (\iota + z)) : (x, y, z) \in U^{\times 3\cdot} \,\},$$

we see that if we have a suitable imFT applicable to the map $(E \sqcap E \sqcap E, E, f)$ at the point $(\mathbf{0}_E, \mathbf{0}_E, \mathbf{0}_E, \mathbf{0}_E)$, we immediately get $\tilde{g} \in C^{\infty}_{\Pi}(\boldsymbol{R})$. Our result [4; Theorem 4.3, pp. 19−20] is such an imFT, but [10; 5.3, p. 45] is not. Omitting further details, we here only mention that as the Bp_2−extension required in [4] one takes the family $\langle\, (\mathrm{F}\, i, \mathrm{f}\, i, \mathrm{id}_{\mathrm{v}} E) : i \in \mathbb{N} \,\rangle$ where $\mathrm{F}\, i = C^{i}_{\mathrm{per}}(\mathbb{R})$ and

$$\mathrm{f}\, i = \{\, (x, y, z, y - z - x \circ (\iota + z)) : (x, y, z) \in U \times U \times (\upsilon_s \mathrm{F}\, i \cap U_1) \,\},$$

and that one gets $\{\, (E \sqcap E \sqcap \mathrm{F}\, i, \mathrm{F}\, i, \mathrm{f}\, i) : i \in \mathbb{N} \,\} \subseteq C^{\infty}_{\Pi}(\boldsymbol{R})$ from [3; Propositions 0.10, 0.11 p. 240] and [4; Proposition 3.1, Theorem 3.6, pp. 15, 17] and the fact $\{\, (G, H, \ell) : G, H \in \mathrm{LCS}(\boldsymbol{R}) \text{ and } \ell \in \mathcal{L}(G, H) \,\} \subseteq C^{\infty}_{\Pi}(\boldsymbol{R})$ by noting that for $\mathrm{f}\, i$ we have the decomposition

$$\begin{aligned}(x, y, z) &\mapsto (\bar{x}, y, z) \mapsto (\bar{x} \circ [\,\mathrm{id}\,, z\,]_{\mathrm{f}}\,, y, z)\\ &= (u, y, z) \mapsto y - z - u = y - z - x \circ (\iota + z),\end{aligned}$$

where the prescription $x \mapsto \bar{x} = \{\, (s, t, x`(s+t)) : s, t \in \mathbb{R} \,\}$ defines a continuous linear map $C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R} \times \mathbb{R})$.

The above (M, γ) of course is nothing but an interpretation of the Lie group $\mathrm{Diff}_{+}\, \mathbb{S}^{1\cdot}$ of orientation preserving smooth diffeomorphism of $\mathbb{S}^{1\cdot}$, constructed so that one may avoid considering the (quite simple) manifold structure of $\mathbb{S}^{1\cdot}$. Considering an arbitrary smooth finite-dimensional paracompact (but not necessarily second countable) manifold M_{bas}, in a manner similar to the above, we can use [4; Theorem 4.3] as a tool when constructing the Lie group $\mathrm{Diff}\, M_{\mathrm{bas}}$. Only the formal details become much more complicated than in the above simple case. It is our intention to give them in 'Mapping families, differentiation, and an application to Lie groups of diffeomorphisms' although some time will be required for the completion of this manuscript. In this connection, one should also note [9] where the same construction problem is treated in a different manner, however, assuming that the topology of M_{bas} is second countable, and still omitting many technical details, although the presentation there generally is unusually detailed.

5 *Remarks.* (a) In the proof of Theorem 11 below, we need a function u whose $i_0{}^{\mathrm{th}}$ canonical (semi)norm is small, and having the absolute value of the $(i_0{}^{+})^{\mathrm{th}}$ derivative large at a given point s_0, with also $u`s_0$ equal to zero. There we can take as u a simple trigonometric function. For function spaces over more general

domains, e.g. finite-dimensional smooth manifolds, we can achieve the same goal by taking instead as u a suitable scalar multiple of a monomial m multiplied by a smooth "bump" function b, pulled back by a chart, and extended by zero.

More precisely, with $N \in I\!N$ and $0 < \delta \le 1$, for $\alpha \in I\!N_{\mathrm{o}}^{N}$ and $\eta \in I\!R^{N}$ letting $\prod \eta = \prod_{i \in N} (\eta\text{\`{}}i)$ and $\eta^{\alpha} = \prod \langle (\eta\text{\`{}}i)^{\alpha\text{\`{}}i} : i \in N \rangle$ and $\alpha! = \prod_{i \in N} ((\alpha\text{\`{}}i)!)$, one takes $m = \mathrm{m}^{\alpha} = \langle \eta^{\alpha} : \eta \in I\!R^{N} \rangle$ and $b = \langle \prod (b_0 \circ (\delta^{-1}\eta)) : \eta \in I\!R^{N} \rangle$, where

$$b_0 = \langle (1 + \exp(((2 - |s|)^2 - 1)^{-1}(2 - |s|)))^{-1} : s \in I\!R \text{ and } 1 < |s| < 3 \rangle \\ \cup ([-1, 1] \times \{1\}) \cup ((I\!R \setminus]-3, 3[) \times \{0\}).$$

Using the Leibniz formula [8; (2), p. 101], with $\eta_0 = N \times \{0\}$, one sees that then $b \cdot m\text{\`{}}\eta_0 = 0$, unless $\alpha = N \times 1_{\bullet}$, and $\partial^{\alpha}(b \cdot m)\text{\`{}}\eta_0 = (\alpha!)^{\bullet}$, and also derives existence of $M \in I\!R^{+}$ independent of δ such that for all $\eta \in I\!R^{N}$ and for $\kappa \le \alpha$ as functions $N \to I\!N_{\mathrm{o}}$ we have the inequality $|\partial^{\kappa}(b \cdot m)\text{\`{}}\eta| \le \delta^{|\alpha| - |\kappa|} M$.

(b) Another aspect of the proof of Theorem 11 below is that a contradiction follows from the assumption that a certain *nonaffine* map $\tilde f$ is "$\mathrm{C}_{\mathrm{B}\Gamma}$" in the sense [10; p. 23], i.e. continuously cb Γ−differentiable $\mu \to \nu$ within $\boldsymbol{E}$ in the sense of our Definition 10 below. More specifically, this contradiction consists of the formulas not $R < R$ and $(*)$ $R < A\,D - M_0\,N_0 \le R$ where $M_0, R \in I\!R_{+}$ with M_0 independent of the varied function u within certain bounds. In $(*)$ we have "$\le$" for all u while "$<$" only for suitably chosen ones. We have $A = |\varphi''\text{\`{}}s_0|$, and hence $A > 0$ for suitably chosen $s_0$ if $\varphi$ is not affine. Further $D = |\mathrm{D}^{i_0 + 1_{\bullet}} u\text{\`{}}s_0|$ and $N_0 = \sup\{ |\mathrm{D}^{l} u\text{\`{}}s| : l \in {i_0}^{+} \text{ and } s \in I\!R \}$. The contradiction is obtained by choosing u so that D becomes large while N_0 remains small.

Suppose that instead of (E, E, f) of Theorem 11 as $\tilde f$ we have the $\tilde h$ of Example 2 above. In the "nonlinear" case where $\partial_3^{2_{\bullet}} \varphi\text{\`{}}P_0 \ne 0$ for some $P_0 = (t_0, \eta_0, \xi_0) \in y \subseteq Q \times I\!R$, we can prove that $\tilde h$ is not "$\mathrm{C}_{\mathrm{B}\Gamma}$" by establishing a corresponding inequality $R < A\,D - M_0\,N_0 \le R$, noting the following complications. We have ${}^{\mathrm{rGat}}D_{GG}\,h\text{\`{}}(z + w)\text{\`{}}w_1 = (\mathbf{0}_E, v_1 - \mathrm{I}\,(\partial_3 \varphi \circ [\,\mathrm{id}, y + v\,]_{\mathrm{f}} \cdot v_1)$ when $z, w, w_1 \in \upsilon_s G$ with $z = (x, y)$ and $w = (\mathbf{0}_E, v)$ and $w_1 = (\mathbf{0}_E, v_1)$. We take $v_1 = Q \times \{1\}$, and for the construction of v we proceed as follows.

For the space G we consider the "canonical" (semi)norms $\langle \|w\|_i : w \in \upsilon_s G \rangle$ where for $w = (u, v) \in \upsilon_s G$ with $e = (1, 1)$ we have

$$\|w\|_i = \sup\{ |\mathrm{D}^{l_1} u\text{\`{}}\eta| + |\partial_1^{k} \partial_2^{l}\, d\,(l_0 \times \{e\})\, v\text{\`{}}\zeta| : \\ l_1, k + l + l_0 \in i^{+} \text{ and } l_0 \in 2_{\bullet} \text{ and } \eta \in I\!R \text{ and } \zeta \in Q \},$$

noting that
$$d\,(l_0 \times \{e\})\, v = \begin{cases} d\,(\emptyset)\, v = v & \text{when } l_0 = \emptyset, \\ d\,(\langle e \rangle)\, v = \partial_1 v + \partial_2 v & \text{when } l_0 = 1_{\bullet}, \end{cases}$$

and that $d\,(\langle e \rangle)\,\mathrm{I}\,v = v$, when $v \in \upsilon_s F$. We take $i = {i_0}^{++}$ and

$$v = \{ (t, \eta, \delta^{-\frac{1}{2}}\, b_0\text{\`{}}(\delta^{-1}(t - t_0)) \cdot (t - t_0)^{i_0 + 1_{\bullet}}) : t \in I \text{ and } \eta \in I\!R \},$$

where $\delta \in I\!R^{+}$ is chosen so that $\|w\|_{i_0}$ will be small while $\partial_1^{i_0 + 1_{\bullet}} v\text{\`{}}(t_0, \eta_0)$ becomes large. For further details, in particular as for the proper order of the various choices, we refer the reader to the proof of Theorem 11 below.

(c) In the situation of Example 3 above, with the provision made there at the end, we obtain $(*)$ as follows. First, by the nonlinearity assumption, whatever $W \in \mathcal{N}_{bh}(z_0, \tau_{\mathrm{rd}}\, G)$ we choose, there always are some $\bar\alpha, \bar\alpha_1 \in A$ and s_0, η_0, ξ_0, T and y with $0 \le s_0 < T$ and $\eta_0 \in \Omega$, and also such that for $P_0 = (s_0, \eta_0, \xi_0)$ and $\zeta_0 = (T^{-1} s_0, \eta_0)$ and $z = (T, y_{\mathrm{ini}\,0}, y)$, we have $z \in W$ and $(T^{-1} s_0, \eta_0, \xi_0) \in \mathrm{J}\, y$

and $\partial_{\bar\alpha}\,\partial_{\bar\alpha_1}\varphi_{\rm old}\grave{}\,P_0 \neq 0$. The contradiction is obtained by considering the required continuity of ${}^{\rm rGat}D_{GG}\,h$ at this point z.

We may assume $|\,\bar\alpha+\bar\alpha_1\,|$ to be the largest possible. For $\zeta=(t,\eta)\in I\!\!R\times I\!\!R^{\,N}$ and $\bar\kappa=(i,\kappa)\in I\!\!N_{\rm o}\times I\!\!N_{\rm o}^{N}$ letting $\zeta^{\,\bar\kappa}=t^{\,i}\,\eta^{\,\kappa}$ and $\bar\kappa!=i!\cdot\kappa!$, note that if also $\bar\nu\in I\!\!N_{\rm o}\times I\!\!N_{\rm o}^{N}$ and $\sigma=\partial^{\,\bar\nu}\langle\,(\zeta-\zeta_0)^{\,\bar\kappa}:\zeta\in\bar\Omega(1)\,\rangle\grave{}\,\zeta_0$, then

$$\sigma=\bar\kappa!^{\bullet}\text{ if }\bar\kappa=\bar\nu\text{, and }\sigma=0\text{ otherwise.}$$

By $\eta_0\in\Omega$, and by the assumption on the boundary conditions, we may choose $\delta_1\in I\!\!R^{+}$ so that for $v_1=\langle\,\prod(b_0\circ(\delta_1^{-1}\,\tau_{\rm rd}(\zeta-\zeta_0)))\cdot(\zeta-\zeta_0)^{\,\bar\alpha_1}:\zeta\in\bar\Omega(1)\,\rangle$ we have $v_1\in v_sF$. We take $i=i_0{}^{++}$ and $\bar\kappa=(i_0{}^{+},{}_N\times 1\boldsymbol{.})$ and $\bar\kappa_1=(i\,,{}_N\times 1\boldsymbol{.})$. We let $R=j\,\|\,v_1\,\|_j$, and we choose δ with $0<\delta\le\delta_1$ so that we get

$$R+M_0\,N_0<T\,|\,\partial_{\bar\alpha}\,\partial_{\bar\alpha_1}\varphi_{\rm old}\grave{}\,P_0\,|\cdot\delta^{-\frac12}\,(\bar\alpha+\bar\kappa)!^{\bullet}\cdot\bar\alpha_1!^{\bullet},$$

for certain $M_0\,,N_0\in I\!\!R^{+}$ which are determined once $\delta_1\,,\bar\alpha\,,\bar\alpha_1\,,\bar\kappa\,,\zeta_0$ have been fixed. With also $v=\big\langle\,\delta^{-\frac12}\prod(b_0\circ(\delta^{-1}\,\tau_{\rm rd}(\zeta-\zeta_0)))\cdot(\zeta-\zeta_0)^{\,\bar\alpha+\bar\kappa}:\zeta\in\bar\Omega(1)\,\big\rangle$, we now take $w=(\mathbf{0}_E\,,v)$ and $w_1=(\mathbf{0}_E\,,v_1)$, to obtain

$$\begin{aligned}
R&<T\,|\,\partial_{\bar\alpha}\,\partial_{\bar\alpha_1}\varphi_{\rm old}\grave{}\,P_0\,|\cdot\delta^{-\frac12}\,(\bar\alpha+\bar\kappa)!^{\bullet}\cdot\bar\alpha_1!^{\bullet}-M_0\,N_0\\
&=|\,\partial_{\bar\alpha}\,\partial_{\bar\alpha_1}(\varphi\grave{}\,T)\circ[\,{\rm id}\,,{\rm J}\,(y+v)\,]_{\rm f}\cdot\partial^{\,\bar\alpha+\bar\kappa}v\cdot\partial^{\,\bar\alpha_1}v_1\grave{}\,\zeta_0\,|-M_0\,N_0\\
&\le|\,\partial^{\,\bar\kappa}(\partial_{\bar\alpha_1}(\varphi\grave{}\,T)\circ[\,{\rm id}\,,{\rm J}\,(y+v)\,]_{\rm f})\cdot\partial^{\,\bar\alpha_1}v_1\grave{}\,\zeta_0\,|\\
&=\big|\,\partial^{\,\bar\kappa}\big(\textstyle\sum_{\,\bar\nu\in A}\partial_{\bar\nu}\,(\varphi\grave{}\,T)\circ[\,{\rm id}\,,{\rm J}\,(y+v)\,]_{\rm f}\cdot\partial^{\,\bar\nu}v_1\big)\grave{}\,\zeta_0\,\big|\\
&=\big|\,\partial^{\,\bar\kappa_1}\big(v_1+{\rm J}_0\big(\textstyle\sum_{\,\bar\nu\in A}\partial_{\bar\nu}\,(\varphi\grave{}\,T)\circ[\,{\rm id}\,,{\rm J}\,(y+v)\,]_{\rm f}\cdot\partial^{\,\bar\nu}v_1\big)\big)\grave{}\,\zeta_0\,\big|\\
&\le|\!|\!|\,{}^{\rm rGat}D_{GG}\,h\grave{}\,(z+w)\grave{}\,w_1\,|\!|\!|_i\le R\,.
\end{aligned}$$

Again, to get a proper proof from the preceding pieces, they have to be put in the right context. For this, we still refer to the proof of Theorem 11 below.

In view of our examples and remarks above, the main importance of Theorem 11 below lies in the idea of its proof, here presented as clearly as possible, free from e.g. blurring differential geometric technicalities. If one wants to get definitely convinced of the theorems [10; 5.2, 5.3, p. 45] not being applicable to a particular (say) differential geometric problem possibly involving a partial differential equation, then one should use the proof of Theorem 11 as a model, and begin to write a proof of length for example some ten pages.

B. The basic concepts and the main result

Since in [10] various loose notational conventions are utilized making matters obscure, we first recall the facts from [10] needed below, reformulated so as to be accordant with the set theoretic notational system we followed in [4] and [7].

6 *Definitions***.** For $E\in{\rm LCS}(\boldsymbol{R})$, let $\mathcal{S}_{\scriptscriptstyle N}E$ be the set of all continuous seminorms on E. For $N\subseteq\mathcal{S}_{\scriptscriptstyle N}E$, we say that N *determines* E iff for every $U\in\mathcal{N}_o\,E$ there are $\varepsilon\in I\!\!R^{+}$ and a finite $N_0\subseteq N$ with $\bigcap\{\,p^{-\iota}``[\,0\,,\varepsilon\,]:p\in N_0\,\}\subseteq U$. We say that Γ is a *calibration* over $\boldsymbol{E}$ iff we have $\boldsymbol{E}\in{\rm LCS}(\boldsymbol{R})^{\,{\rm dom}\,\boldsymbol{E}}$ with $\Gamma\subseteq\prod_{\mathsf{c}}\{\,(\nu,\mathcal{S}_{\scriptscriptstyle N}E):(\nu,E)\in\boldsymbol{E}\,\}$ such that $\bigcup\Gamma``\{\nu\}$ determines E whenever $(\nu,E)\in\boldsymbol{E}$.

One easily sees that N determines a given $E\in{\rm LCS}(\boldsymbol{R})$ iff every $p\in\mathcal{S}_{\scriptscriptstyle N}E$ has some finite $N_0\subseteq N$ and $M\in I\!\!R^{+}$ with $p\grave{}\,x\le\sup\{\,M\,r:\exists\,q\,;\,(x\,,r)\in q\in N_0\,\}$ for all $x\in v_sE$. Note also that if Γ is a calibration over $\boldsymbol{E}$, then Γ and $\boldsymbol{E}$ are necessarily small families. It follows that the speech for example in [10; Example 1, p. 4] of having a calibration over the class of all normable spaces does not make sense in our set theory. In [10; p. 3], one refers by the term "seminorm map"

to the elements of the product set $\prod_{\mathbf{c}}\{\,(\nu\,,\mathcal{S}_N\,E\,):(\nu\,,E\,)\in \boldsymbol{E}\,\}$ which is empty (in our set theory) if $\boldsymbol{E}$ is a large family of locally convex spaces.

Arbitrarily fixing any two-element set I_0, for example taking as I_0 the cardinal number $2. = \{\emptyset\,,1.\} = \{\emptyset\,,\{\emptyset\}\}$, for the purpose of this note it would suffice to consider only calibrations over $\boldsymbol{E}$ with $\operatorname{dom}\boldsymbol{E} = I_0$.

7 *Definitions* (Gateaux differentiability)**.**

$$\begin{aligned}
{}^{\mathrm{rGat}}\mathsf{D}_{E\,F}\,f = \{\,(x\,,\ell\,) : {} & E\,,F \in \mathrm{LCS}\,(\boldsymbol{R}) \text{ and } f \in \upsilon_s F^{\,\operatorname{dom} f} \text{ and} \\
& \operatorname{dom} f \in \mathcal{N}_{bh}(x\,,\tau_{\mathrm{rd}}\,E\,) \text{ and } \ell \in \mathcal{L}\,(E\,,F\,) \text{ and} \\
& \forall\, u \in \upsilon_s E\,, V \in \mathcal{N}_o\,F\,;\ \exists\,\delta \in I\!R^+\,;\ \forall\, t \in I\!R\,; \\
& 0 < |\,t\,| < \delta \Rightarrow (t^{-1}(f\,\grave{}\,(x+t\,u)_{\mathrm{svs}\,E} - f\,\grave{}\,x)_{\mathrm{svs}\,F} - \ell\,\grave{}\,u)_{\mathrm{svs}\,F} \in V\,\}\,, \\
\tilde{f}\,'(x) = \bigcap\{\,{}^{\mathrm{rGat}}\mathsf{D}_{E\,F}\,f\,\grave{}\,x : {} & \tilde{f} = (E\,,F\,,f\,)\,\}\,.
\end{aligned}$$

By a real *Gateaux differentiable* map we understand any $\tilde{f} = (E\,,F\,,f\,)$ such that $E\,,F \in \mathrm{LCS}\,(\boldsymbol{R})$ and $f \in \upsilon_s F^{\,\operatorname{dom} f}$ and $\operatorname{dom} f \subseteq \operatorname{dom}{}^{\mathrm{rGat}}\mathsf{D}_{E\,F}\,f$.

It follows that if $(E\,,F\,,f\,)$ is Gateaux differentiable, then $\operatorname{dom} f \in \tau_{\mathrm{rd}}\,E$ since for every $x \in \operatorname{dom} f$ we have $\operatorname{dom} f \in \mathcal{N}_{bh}(x\,,\tau_{\mathrm{rd}}\,E\,)$. For all $E\,,S$ generally having $\mathrm{Of}\ E_{\mathrm{sub\,tvs}}\,S = E_{/\,S} = (\sigma_{\mathrm{rd}}\,E_{|\,S}\,,\tau_{\mathrm{rd}}\,E \cap S\,)$, we consider the Fréchet space

$$C^{\infty}_{\mathrm{per}}(I\!R) = \mathrm{Of}\ C^{\infty}(I\!R)_{\mathrm{sub\,tvs}}\{\,x : \forall\, s \in I\!R\,;\ x\,\grave{}\,s = x\,\grave{}\,(s+1)\,\}$$

of smooth $1-$periodic functions $I\!R \to I\!R$ in the following

8 Lemma. *Let* $E = C^{\infty}_{\mathrm{per}}(I\!R)$ *and* $f = \langle\,\varphi \circ x : x \in \upsilon_s E\,\rangle$ *where* φ *is a smooth function* $I\!R \to I\!R$. *Then* $(E\,,E\,,f\,)'(x) = \langle\,\varphi' \circ x \cdot v : v \in \upsilon_s E\,\rangle$ *for all* $x \in \upsilon_s E$.

Proof. Let $\varphi_1 = \{\,(s\,,t\,,\varphi\,\grave{}\,t) : s\,,t \in I\!R\,\}$ and $f_1 = \langle\,\varphi \circ x : x \in \upsilon_s F\,\rangle$, where $F = C^{\infty}(I\!R)$, and first consider the map $\tilde{f}_1 = (F\,,F\,,f_1)$. Since we can decompose it $F \to C^{\infty}(I\!R \times I\!R) \sqcap F \to F$ by $x \mapsto (\varphi_1\,,x) \mapsto \varphi_1 \circ [\,\mathrm{id}\,,x\,] = \varphi \circ x$, where the first factor is a continuous affine map, hence smooth, and the second is smooth by [4; Theorem 3.6, p. 17], by the chain rule [3; Proposition 0.11 p. 240] we get $\tilde{f}_1 \in C^{\infty}_{\Pi}(\boldsymbol{R})$. Since E is a (sequentially) closed topological linear subspace of F, and since we have $f = f_1\,|\,\upsilon_s E$ with $\operatorname{rng} f \subseteq \upsilon_s E$, the assertion of the lemma follows from [4; Proposition 3.1, Remarks 3.7(b), pp. 15, 17–18] in conjunction with elementary set theoretic manipulations applied to Definitions 7 above. $\square$

9 *Remarks.* Given a calibration Γ over $\boldsymbol{E}$ with $(\mu\,,E\,),(\nu\,,F\,) \in \boldsymbol{E}$ and $L = \mathcal{L}_{\mathrm{b}\,\Gamma}(\mu\,,\nu)_{\boldsymbol{E}}$, then L is the unique normable locally convex space with $\sigma_{\mathrm{rd}}\,L$ a vector substructure of $\sigma_{\mathrm{rd}}\,F^{\,\upsilon_s E\,]_{\mathrm{vs}}}$ and $\upsilon_s L = \nu^{\,-\iota\,``}I\!R_+$ and $\nu^{\,-\iota\,``}[\,0\,,1\,] \in \mathcal{B}_s\,L \cap \mathcal{N}_o\,L$ when we let $\nu = \langle\,\sup\{\,\boldsymbol{p}\,\grave{}\,\nu\,\grave{}\,(\ell\,\grave{}\,v) : \boldsymbol{p} \in \Gamma \text{ and } \boldsymbol{p}\,\grave{}\,\mu\,\grave{}\,v \le 1\,\} : \ell \in \mathcal{L}\,(E\,,F\,)\,\rangle$. Here we call $\nu\,|\,\upsilon_s L$ the *canonical* Γ*-norm* $\mu \to \nu$ over $\boldsymbol{E}$. In [10], the space L is imprecisely denoted by "$\mathrm{L}_{\mathrm{B}\Gamma}(E,F)$". If also F is sequentially complete, then L is Banachable, and $(\sigma_{\mathrm{rd}}\,L\,,\nu\,|\,\upsilon_s L)$ is a (normed) Banach space, cf. [10; 2.3, p. 6].

In view of [10; 5.2, 5.3, p. 45], the "continuous $\mathrm{B}\Gamma-$differentiability" (or being $\mathrm{C}_{\mathrm{B}\Gamma}$) should be a most important concept in [10]. Despite this, on page 23 there, its definition is only vaguely sketched, and we replace this concept by the generally weaker one given in the following

10 *Definition.* A real Gateaux differentiable map $\tilde{f} = (E\,,F\,,f\,)$ we say to be *continuously* cb Γ*-differentiable* $\mu \to \nu$ within $\boldsymbol{E}$ iff Γ is a calibration over $\boldsymbol{E}$ with $(\mu\,,E\,),(\nu\,,F\,) \in \boldsymbol{E}$ such that for $L = \mathcal{L}_{\mathrm{b}\,\Gamma}(\mu\,,\nu)_{\boldsymbol{E}}$ and for ν the canonical $\Gamma-$

norm $\mu \to \nu$ over $\boldsymbol{E}$, we have $\operatorname{rng}{}^{\mathrm{rGat}}D_{EF}\, f \subseteq \upsilon_s L$ and also for $x \in \operatorname{dom} f$ and $\boldsymbol{p} \in \Gamma$ and $\varepsilon \in I\!R^+$ there is $\delta \in I\!R^+$ such that for all u, y we have the implication

$$\boldsymbol{p}`\mu`u < \delta \text{ and } y = x + u \in \operatorname{dom} f \;\Rightarrow\; \nu`(\tilde{f}'(y) - \tilde{f}'(x)) < \varepsilon\,.$$

It follows from [10; 2.6, p. 29] that if $\operatorname{dom} f$ is convex or if Γ is such that for every $\boldsymbol{p} \in \Gamma$ and $x \in \operatorname{dom} f$ there is $\delta \in I\!R^+$ with $\{\, x + u : \boldsymbol{p}`\mu`u < \delta \,\} \subseteq \operatorname{dom} f$, cf. [10; pp. 18–19], then $\tilde{f}$ is continuously cb Γ-differentiable $\mu \to \nu$ within $\boldsymbol{E}$ iff it is "$\mathrm{C_{B\Gamma}}$". Consequently, noting that E is locally convex, if $\tilde{f}$ is continuously cb Γ-differentiable $\mu \to \nu$ within $\boldsymbol{E}$, for every $x \in \operatorname{dom} f$ there is U with $x \in U$ such that $(E, F, f\,|\,U)$ is "$\mathrm{C_{B\Gamma}}$". In Remarks 5 (c) above, we gave the basic ingredients for the proof of [10; 5.2, p. 45] not being applicable to the map $(G, G, h\,|\,W)$ anyhow one chooses W with $z_0 \in W \in \tau_{\mathrm{rd}}\, G$.

11 Theorem. *Let $E = C^\infty_{\mathrm{per}}(I\!R)$ and $f = \langle\, \varphi \circ x : x \in \upsilon_s E \,\rangle$ where φ is a smooth function $I\!R \to I\!R$. If (E, E, f) is continuously cb Γ-differentiable $\mu \to \nu$ within $\boldsymbol{E}$, then there are $\alpha, \beta \in I\!R$ with the property that $\varphi = \langle\, \alpha\, t + \beta : t \in I\!R \,\rangle$.*

Proof. Let $\tilde{f} = (E, E, f)$. Under the premise, arbitrarily fixing $s_0 \in I\!R$, it suffices to prove indirectly that $\varphi''`s_0 = 0$. To get a contradiction, let $\varphi''`s_0 \neq 0$ and consider $x = I\!R \times \{s_0\}$. Let ν be the canonical Γ-norm $\mu \to \nu$ over $\boldsymbol{E}$, and put $M = \nu`(\tilde{f}'(x)) + 1$. Since $\bigcup \Gamma``\{\mu\}$ determines E, we have $\Gamma \neq \emptyset$, and so we can pick $\boldsymbol{p}_0 \in \Gamma$. Taking this $\boldsymbol{p}_0$ in place of $\boldsymbol{p}$ and $\varepsilon = 1$ in Definition 10 above, there is $\delta \in I\!R^+$ such that for $\boldsymbol{p}_0`\mu`u < \delta$ we have $\nu`(\tilde{f}'(x+u) - \tilde{f}'(x)) < 1$, whence further $\nu`(\tilde{f}'(x+u)) \leq \nu`(\tilde{f}'(x)) + 1 = M$. Consequently, noting also Lemma 8 above, for all $\boldsymbol{p} \in \Gamma$ we have

(1) $\quad \boldsymbol{p}`\nu`(\varphi' \circ (x+u) \cdot v) \leq M\,(\boldsymbol{p}`\mu`v)$ when $\boldsymbol{p}_0`\mu`u < \delta$ and $v \in \upsilon_s E$.

Take $v = I\!R \times \{1\}$. Letting $\|z\|_i = \sup\{\, |\mathrm{D}^l z`s| : l \in i^+ \text{ and } s \in I\!R \,\}$, then also $\{\, \langle\, \|z\|_i : z \in \upsilon_s E \,\rangle : i \in I\!N_{\mathrm{o}} \}$ determines E. Hence, there is an even $i_0 \in I\!N$ such that we have the implication

(2) $\quad i_0\, \|u\|_{i_0} < 1 \;\Rightarrow\; \boldsymbol{p}_0`\mu`u < \delta$ for all $u \in \upsilon_s E$.

Letting $i = i_0{}^+$, we have i odd. Now, there are a finite $P \subseteq \Gamma$ and $M_1 \in I\!R^+$ such that for

$$q = \langle\, \sup\{\, M_1 r : \exists\, \boldsymbol{p} \in P\,;\, (z, r) \in \boldsymbol{p}`\nu \,\} : z \in \upsilon_s E \,\rangle \quad \text{and}$$
$$p = \langle\, \sup\{\, M M_1 r : \exists\, \boldsymbol{p} \in P\,;\, (z, r) \in \boldsymbol{p}`\mu \,\} : z \in \upsilon_s E \,\rangle\,,$$

we have $\|z\|_i \leq q`z$ for all $z \in \upsilon_s E$. Having $p \in \mathcal{S}_N E$, there further is $j \in I\!N_{\mathrm{o}}$ with $p`z \leq j\, \|z\|_j$ for all $z \in \upsilon_s E$. Using these and $j\, \|v\|_j = j\, 1 = j^\bullet$, from (1) and (2) we obtain

(3) $\quad \|\varphi' \circ (x+u)\|_i \leq j^\bullet$ for all $u \in \upsilon_s E$ with $i_0\, \|u\|_{i_0} < 1$.

Next, noting that $|r + t| \geq |r| - |t|$ for $r, t \in I\!R$, and utilizing the quite combinatorial idea in the proof of [6; Proposition 10, pp. 6–7], one deduces existence of $M_2 \in I\!R^+$ such that for all $u \in \upsilon_s E$ and $s \in I\!R$ with $\|u\|_{0.} \leq 1$ we have

(4) $\quad |\mathrm{D}^i(\varphi' \circ (x+u))`s| \geq |\varphi'' \circ (x+u) \cdot \mathrm{D}^i u`s| - M_2\,(1 + \|u\|_{i_0})^i$.

With $A = |\varphi''`s_0| \in I\!R^+$, now choosing $n \in Z\!\!\!Z^+$ so that we have the inequalities $j^\bullet < A\,(2\pi n)^{\frac12} - M_2\,\bigl(1 + (2\pi n)^{-\frac12}\bigr)^i$ and $i_0\,(2\pi n)^{-\frac12} < 1$, we take

$$u = \bigl\langle\, (2\pi n)^{-i_0{}^\bullet - \frac12} \sin(2\pi n\,(s - s_0)) : s \in I\!R \,\bigr\rangle\,.$$

Then we have $i_0\, \|u\|_{i_0} = i_0\,(2\pi n)^{-\frac12} < 1$, whence recalling that i is odd, by (3) and (4) we obtain

$$
\begin{aligned}
j^{\bullet} &< A\,(2\,\pi\,n)^{\frac{1}{2}} - M_2\left(1 + (2\,\pi\,n)^{-\frac{1}{2}}\right)^i \\
&= |\,\varphi''\,`s_0\,|\,|\,\mathrm{D}^i u\,`s_0\,| - M_2\,(1 + \|\,u\,\|_{i_0})^i \\
&= |\,\varphi'' \circ (x+u) \cdot \mathrm{D}^i u\,`s_0\,| - M_2\,(1 + \|\,u\,\|_{i_0})^i \\
&\le |\,\mathrm{D}^i(\varphi' \circ (x+u))\,`s_0\,| \le \|\,\varphi' \circ (x+u)\,\|_i \le j^{\bullet}, \text{ a contradiction.} \qquad \square
\end{aligned}
$$

Note that basic idea in the preceding proof is the same which we already utilized when establishing [5; Propositions 3, 5].

12 *Remark***.** Fixing a calibration is required also for the inFT [11; 8.4, p. 457] but this result has a nature quite different from that of the IFTs in [10; p. 45]. It seems that [11; 8.4] has genuine applications although quite an amount of work is required for the verification of the premise. Here, by a *genuine* application of an IFT generalizing the corresponding Banach space theorem we mean an application not covered by the classical theorem.

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